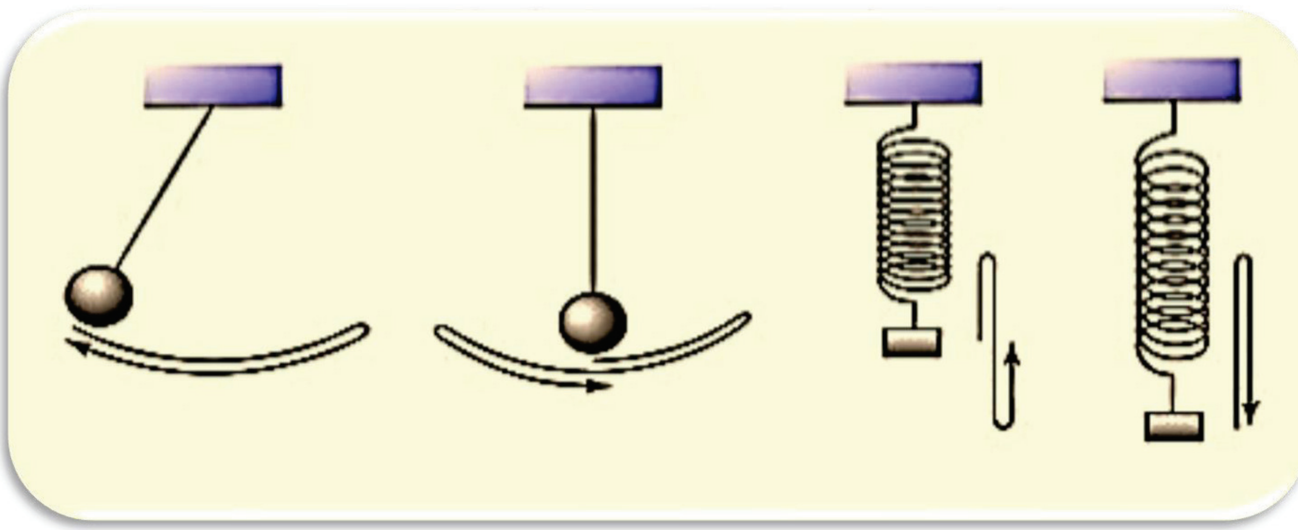
Oscillations



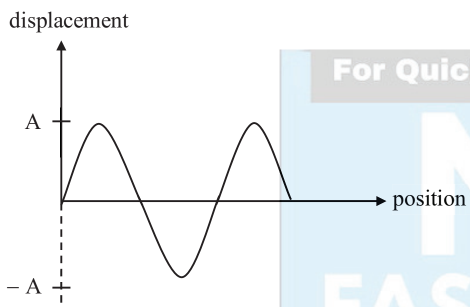
Oscillations

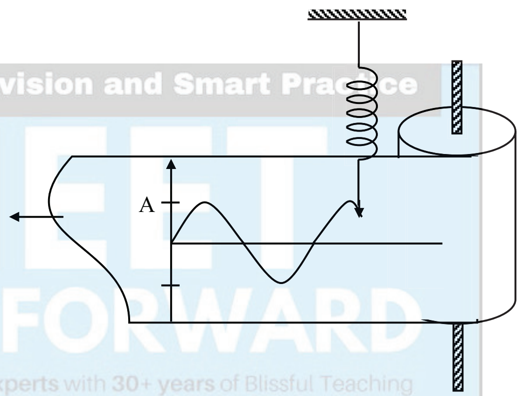
A repeated motion at equal interval of time is called "periodic motion". If a particle moves back and forth (or to and fro) over the same path, then its motion is said to be oscillatory or vibratory.
When oscillatory motion of a particle can be expressed in terms of sine and cosine functions, it is said to be a harmonic motion. All harmonic motions are oscillatory.

Simple harmonic motion is a special case of harmonic motion in which restoring force acting on the particle is directly proportional to its displacement from the equilibrium position, i.e., $F \propto x$
or $F = -kx$, where k is called the force constant. Negative sign shows that the force is always directed towards the equilibrium position.

$\therefore\ F = ma = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$ ...(1)

This equation is a second order linear homogeneous differential equation. It is called the differential equation of simple harmonic motion. The figure shows the representation of a SHM.





Conditions of simple harmonic motion

1. There must be a stable equilibrium position.
2. The motion must be symmetrical about the equilibrium position.
3. The restoring force must be proportional to the displacement and should be opposite to direction of displacement.
4. Velocity should be a continuous function of time.

Basic terms related to SHM

1. Amplitude (A)
   It is the maximum displacement of an oscillating body from its equilibrium position or mean position.
2. Period (T)
   It is the time taken by an oscillating particle to complete one oscillation.
3. Frequency (f)
   It is the number of oscillations made in one second by the oscillating particle.
4. Angular frequency (ω)
   It is the rate of change of angle. $\omega = 2\pi/T = 2\pi f$

Equation of SHM

Solutions of the differential equation (1) are the equations of SHM.
In general, the equation of motion may be represented by any of the following equations.
$x = A \sin(\omega t + \phi)$
$x = A \cos(\omega t + \phi)$

$x = a \sin \omega t + b \cos \omega t$

We use $x = A \sin(\omega t + \phi)$ as the general equation of motion, where ϕ is called the phase constant and $(\omega t + \phi)$, the phase of the particle.

Simple harmonic motion as a projection of circular motion

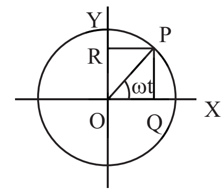
Consider a particle P moving on a circle of radius A with a constant angular speed ω. Let us take the centre of the circle as the origin and two perpendicular diameters as the X and Y-axis at time t. Drop perpendicular PQ on X-axis and PR on Y-axis. The x and y-coordinates of the particle at time t are

$x = OQ = OP \cos \omega t$

or, $x = A \cos \omega t$ … (1)

and $y = OR = OP \sin \omega t$

or, $y = A \sin \omega t$ … (2)



Equation (1) shows that the foot of perpendicular Q executes a simple harmonic motion on the X-axis. The amplitude is A and the angular frequency is ω. Similarly, equation (2) shows that the foot of perpendicular R executes a simple harmonic motion on the Y-axis. The amplitude is A and the angular frequency is ω. The phases of the two simple harmonic motions differ by $\frac{\pi}{2}$ [remember $\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$

Thus, the projection of uniform circular motion on a diameter of the circle is a simple harmonic motion.

Displacement, velocity and acceleration as a function of time

Suppose we start measuring time when the body is in equilibrium.

Then $\phi = 0$. The instantaneous velocity of the body,

v is given by $v = \frac{dx}{dt} = \omega A \cos \omega t$.

The figure shows the plot of velocity as a function of time.

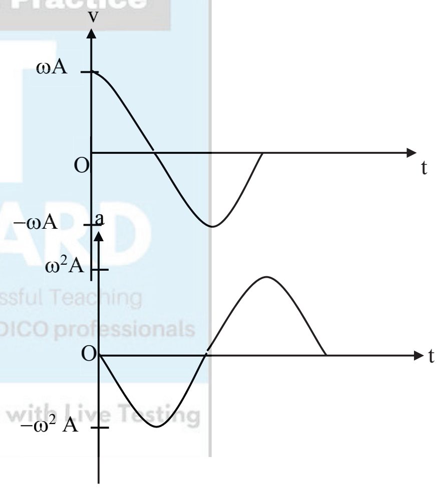
$$v = \frac{dx}{dt} = \omega A \cos \omega t \quad \text{or} \quad v = \omega A \sin\left(\omega t + \frac{\pi}{2}\right)$$

The acceleration of the body is given by,

$$a = \frac{dv}{dt} = -\omega^2 A \sin \omega t$$

$$a = \omega^2 A \sin(\omega t + \pi)$$

The figure shows a plot of acceleration v/s time.



1. The maximum displacement is A, the maximum velocity is ωA and the maximum acceleration is $\omega^2 A$.
2. The velocity is $\frac{\pi}{2}$ ahead of the displacement and the acceleration is $\frac{\pi}{2}$ ahead of the velocity or π ahead of the displacement.

Velocity and acceleration as a function of displacement

$$\mathbf{a = -\omega^2 x}$$

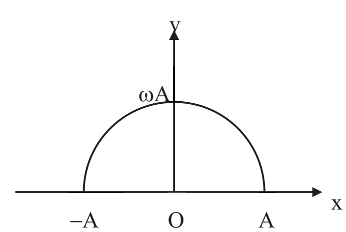
Since $a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$, we can write,

$$v\frac{dv}{dx} = -\omega^2 x \quad \therefore vdv = -\omega^2 x dx$$

On integrating, we get, $\frac{v^2}{2} = -\omega^2 \frac{x^2}{2} + C$ (where C is a constant)

At $x = 0$, $v = v_{max} = \omega A$ $\therefore C = \frac{\omega^2 A^2}{2}$

$$\therefore v = \omega\sqrt{A^2 - x^2}$$



Illustrations

1. A particle executes simple harmonic motion with a period of 3s and amplitude 5 cm. At = 0, it is at position $x = 2.5$ cm going along the positive x-axis. The equation for the displacement of the particle at any instant t is given by

(A) $x = 5\sin\left(\frac{\pi}{3}t + \frac{\pi}{3}\right)$ (B) $x = 5\sin\left(\frac{\pi}{3}t + \frac{2\pi}{3}\right)$

(C) $x = 5\sin\left(\frac{2\pi}{3}t + \frac{\pi}{3}\right)$ (D) $x = 5\sin\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right)$

Ans (D)

In general, the equation for displacement as function of time is given by $x = A \sin(\omega t + \theta)$

Here $A = 5$ cm and $\omega = \frac{2\pi}{t} = \frac{2\pi}{3}$

at $t = 0$; $x = A \sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{A} = \sin^{-1}\frac{2\cdot 5}{5} = \frac{\pi}{6}$

$\therefore$ The equation is $x = 5\sin\left(\frac{2\pi}{3}t + \frac{\pi}{6}\right)$

2. The simple harmonic motion of a particle is represented by an equation $x = 10\sin\left(10t + \frac{\pi}{3}\right)$ and the motion starts at $t = 0$. When does the particle first come to rest?

(A) $\frac{\pi}{30}$ s (B) $\frac{\pi}{60}$ s (C) $\frac{\pi}{40}$ s (D) $\frac{\pi}{80}$ s

Ans (B)

$x = 10 \sin\left(10t + \frac{\pi}{3}\right)$; $v = \frac{dx}{dt} = 100 \cos\left(10t + \frac{\pi}{3}\right)$

$v = 0$ when $10t + \frac{\pi}{3} = \frac{\pi}{2} \Rightarrow 10t = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{60}$ s

3. A particle executes simple harmonic motion with an amplitude 1.414 m and a time period of 24 s. The time taken by the particle to cover half the amplitude is

(A) 4 s (B) 3 s (C) 2 s (D) 1 s

Ans (C)

$x = A\sin\omega t$; $\frac{A}{2} = A \sin\omega t' \Rightarrow \omega t' = \frac{\pi}{6} \Rightarrow \frac{2\pi}{T}t' = \frac{\pi}{6}3$

$t' = \frac{T}{12} = 2$ s

4. A particle is performing SHM in a straight line. If the velocity of the particle are v_1 and v_2 at displacements x_1 and x_2 respectively, the angular frequency of the particle is

(A) $\sqrt{\frac{v_1^2 + v_2^2}{x_2^2 + x_1^2}}$ (B) $\sqrt{\frac{v_1^2 - v_2^2}{x_2^2 + x_1^2}}$ (C) $\sqrt{\frac{v_1^2 - v_2^2}{x_1^2 - x_2^2}}$ (D) $\sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$

Ans (D)

$v = \omega\sqrt{A^2 - x^2}$ $\therefore$ $v_1 = \omega\sqrt{A^2 - x_1^2}$ and $v_2 = \omega\sqrt{A^2 - x^2}$

or $v_1^2 = \omega^2 (A^2 - x_1^2)$ and $v_2^2 = \omega^2(A^2 - x_2^2)$

$\therefore$ $v_1^2 - v_2^2 = (x_2^2 - x_1^2)\omega^2 \Rightarrow \omega^2 = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2} \Rightarrow \omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$